

On pathwise counterparts of Doob's maximal inequalities¹

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Abstract

In this paper, we present pathwise counterparts of Doob's maximal inequalities (on the probability of exceeding a level) for submartingales and supermartingales.

Recently a new method of proving martingale inequalities became popular. Namely, they are derived (and thus often improved) from elementary *deterministic* inequalities. As typical examples, let us mention Doob's maximal L^p - and $L \log L$ -inequalities [1] and the Burkholder–Davis–Gundy inequality [3]. Deterministic inequalities have a natural interpretation in terms of robust hedging of options, see [10], which served as the impetus for their appearance. We also mention the papers [12], [4], [2], [5], [9], [11], dealing with similar problems.

The purpose of this note is to present elementary pathwise counterparts of Doob's maximal inequalities on the probability of exceeding a level. Substituting a trajectory of a stochastic process in our inequality and taking expectations, we obtain Doob's inequality for supermartingales and submartingales, see [7, Chapter VII, Theorem 3.2], due to the fact that one of the terms in the inequalities can be dropped if the process is a supermartingale (in the case of the first inequality) or a submartingale (in the case of the second one). We also show that the pathwise counterparts of Doob's maximal L^p - and $L \log L$ -inequalities from the paper [1] can be obtained by integration from our inequalities.

Let $x = (x_0, \dots, x_n)$ be a vector of real numbers. Put

$$\bar{x}_k = \max \{x_0, \dots, x_k\}, \quad k = 0, \dots, n; \quad \Delta x_k = x_k - x_{k-1}, \quad k = 1, \dots, n.$$

The symbol $\mathbb{1}_A$ stands for the indicator function that is 1 on a set A and 0 outside A .

Theorem 1. *For any $\lambda \in \mathbb{R}$,*

$$\lambda \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} \leq x_0 \wedge \lambda + \sum_{k=1}^n \mathbb{1}_{\{\bar{x}_{k-1} < \lambda\}} \Delta x_k - x_n \mathbb{1}_{\{\bar{x}_n < \lambda\}}, \quad (1)$$

$$\lambda \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} \leq -(x_0 - \lambda) \mathbb{1}_{\{x_0 \geq \lambda\}} - \sum_{k=1}^n \mathbb{1}_{\{\bar{x}_{k-1} \geq \lambda\}} \Delta x_k + x_n \mathbb{1}_{\{\bar{x}_n \geq \lambda\}}. \quad (2)$$

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The proof of inequalities (1) and (2) reduces to their trivial verification in three cases: $\bar{x}_n < \lambda$, $x_0 \geq \lambda$, $\bar{x}_{j-1} < \lambda \leq x_j$ ($j = 1, \dots, n$). Moreover, in the first two cases both inequalities are equalities and, in the third case, the difference of the right-hand and left-hand sides equals $x_j - \lambda$ in both inequalities.

Now let $X = (X_k)_{k=0,1,\dots,n}$ be a stochastic process given on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0,1,\dots,n}, \mathbf{P})$.

Corollary 1. *Let $\lambda \in \mathbb{R}$.*

(i) *If X is a supermartingale, then*

$$\lambda \mathbf{P}(\bar{X}_n \geq \lambda) \leq \mathbf{E}(X_0 \wedge \lambda) - \int_{\{\bar{X}_n < \lambda\}} X_n d\mathbf{P}. \quad (3)$$

(ii) *If X is a submartingale, then*

$$\lambda \mathbf{P}(\bar{X}_n \geq \lambda) \leq -\mathbf{E}[(X_0 - \lambda) \mathbb{1}_{\{X_0 \geq \lambda\}}] + \int_{\{\bar{X}_n \geq \lambda\}} X_n d\mathbf{P}. \quad (4)$$

Inequalities (3) and (4) slightly improve the original inequalities due to Doob [7, Chapter VII, Theorem 3.2] that are obtained if we replace $\mathbf{E}(X_0 \wedge \lambda)$ by $\mathbf{E}(X_0)$ in (3) and drop the first term $-\mathbf{E}[(X_0 - \lambda) \mathbb{1}_{\{X_0 \geq \lambda\}}]$ on the right in (4).

Now let all x_0, \dots, x_n be nonnegative, $p > 1$, $q = p/(p-1)$. Then the following relations hold, where the first inequality follows from (2) and the second one follows from the inequality $ab \leq a^p/p + b^q/q$ ($a, b \geq 0$):

$$\begin{aligned} \bar{x}_n^p &= p \int_0^\infty \lambda^{p-1} \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} d\lambda \\ &\leq p \int_0^\infty \lambda^{p-2} x_n \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} d\lambda - p \int_0^\infty \lambda^{p-2} (x_0 - \lambda) \mathbb{1}_{\{x_0 \geq \lambda\}} d\lambda - p \sum_{k=1}^n \int_0^\infty \lambda^{p-2} \mathbb{1}_{\{\bar{x}_{k-1} \geq \lambda\}} \Delta x_k d\lambda \\ &= qx_n \bar{x}_n^{p-1} - qx_0^p + x_0^p - q \sum_{k=1}^n \bar{x}_{k-1}^{p-1} \Delta x_k \\ &\leq \frac{\bar{x}_n^p}{q} + \frac{q^p x_n^p}{p} - (q-1)x_0^p - q \sum_{k=1}^n \bar{x}_{k-1}^{p-1} \Delta x_k. \end{aligned}$$

Therefore,

$$\bar{x}_n^p \leq q^p x_n^p - qx_0^p - qp \sum_{k=1}^n \bar{x}_{k-1}^{p-1} \Delta x_k. \quad (5)$$

Inequality (5) was obtained in [1]. It implies the following minor generalization of Doob's maximal L^p -inequality: if X is a nonnegative submartingale and $\mathbf{E}X_n^p < \infty$, then

$$\mathbf{E}[\bar{X}_n^p] \leq q^p \mathbf{E}[X_n^p] - q \mathbf{E}[X_0^p],$$

see [6].

In conclusion let us consider the case $p = 1$. Assume additionally that $x_0 > 0$. Then the following relations hold, where the first inequality follows from (2) and the second one follows from the inequality $a \log b \leq a \log a + e^{-1}b$ ($a \geq 0, b > 0$):

$$\begin{aligned}
\bar{x}_n &= x_0 + \int_{x_0}^{\infty} \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} d\lambda \\
&\leq x_0 + x_n \int_{x_0}^{\infty} \lambda^{-1} \mathbb{1}_{\{\bar{x}_n \geq \lambda\}} d\lambda - \sum_{k=1}^n \Delta x_k \int_{x_0}^{\infty} \lambda^{-1} \mathbb{1}_{\{\bar{x}_{k-1} \geq \lambda\}} d\lambda \\
&= x_0 + x_n \log \bar{x}_n - x_n \log x_0 - \sum_{k=1}^n \log(\bar{x}_{k-1}/x_0) \Delta x_k \\
&\leq x_0 + x_n \log(x_n/x_0) + e^{-1} \bar{x}_n - \sum_{k=1}^n \log(\bar{x}_{k-1}/x_0) \Delta x_k.
\end{aligned}$$

Therefore,

$$\bar{x}_n \leq \frac{e}{e-1} \left[x_0 + x_n \log(x_n/x_0) - \sum_{k=1}^n \log(\bar{x}_{k-1}/x_0) \Delta x_k \right] \quad (6)$$

$$= \frac{e}{e-1} \left[x_0(1 - \log x_0) + x_n \log x_n - \sum_{k=1}^n \log \bar{x}_{k-1} \Delta x_k \right]. \quad (7)$$

This inequality with the right-hand side as in (7) is proved in [1]. Rewriting it in the form (6) allows us to drop the last term with the sum after substituting a nonnegative submartingale X for x and taking expectations. At the same time, in general, the last term in (7) has an indefinite sign after substituting a submartingale and taking expectations if X is not a martingale. More precisely, if X is a nonnegative martingale such that $\mathbb{E}[X_n \log X_n] < +\infty$, then the inequality

$$\mathbb{E}[\bar{X}_n] \leq \frac{e}{e-1} \{ \mathbb{E}[X_0(1 - \log X_0)] + \mathbb{E}[X_n \log X_n] \} \quad (8)$$

holds. It is easy to see that this is not true in general if X is a (strictly positive) submartingale: it is enough to put $n = 1$, $X_0 = \varepsilon$, where $\varepsilon > 0$ is small enough, and $X_1 = 1$. In other words, inequality (Doob- L^1) in the statement of Theorem 1.1 in [1] is valid for nonnegative martingales and is not valid for submartingales as is stated in this theorem. Nevertheless, the following improvement of Doob's maximal $L \log L$ -inequality is true: for any nonnegative submartingale X ,

$$\mathbb{E}[\bar{X}_n] \leq \frac{e}{e-1} \{ 1 + \mathbb{E}[X_n \log X_n] \}. \quad (9)$$

For martingales, (9) follows from (8). If X is a submartingale, then the inequality follows from (9) applied to the martingale $Y_k = \mathbb{E}[X_n | \mathcal{F}_k]$, since, clearly, $Y_n = X_n$ and $\bar{Y}_n \geq \bar{X}_n$. Recall that, in Doob's maximal $L \log L$ -inequality, unlike (9), there appears \log^+ instead of \log , and that the constant $e/(e-1)$ in (9) is optimal, see [8].

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